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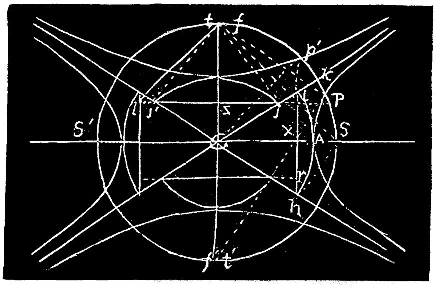
FOUR NEW THEOREMS RELATING TO CONJUGATE HYPERBOLAS.

By REV. ALAN S. HAWKESWORTH, Philadelphia, Pa.

Theorem 1. *The portions of the directrices of conjugate hyperbolas, cut off between their common asymptotes, are equal, each to the other* [*i. e.*, $hh'=jj'$]*; and subtend supplementary angles at their respective foci* [*i. e.*, $\angle hsh'+\angle jfj'=180°$].

Thus, the four quadrilaterals [$shch'$, and $fjcj'$], whose diagonals are the four directrices and semi-focal distances, are equal in all respects and proportions; but reverse in direction [$ch=jf$; $sh=cj$; $\angle hsh'=\angle jcj'$; and $\angle hch'=\angle jfj'$]. So that the focal distance of each directrix is equal to the distance from the center, C, of the curve of its conjugate directrix; and vice versa. [$xs=cz$; and $zf=cx$].



Join sh, sh'; fj and fj'; and draw AK and BK, the tangents at the apeci A and B. Then $Cs=CK=Cf$; $CA=Ch$; and $CB=Cj$. Therefore $CK:CA::cs:ch$; and so Chs is a right angled triangle; and equal to CKA. And similarly, $CK:CB::cf:cj$; so that cjf is also a right angled triangle; and equal to CKB; and therefore also to CKA, and Chs.

And, as the right angled triangles Cjf and Chs are equal in all respects and proportions, so also are their doubles, the quadrilaterals $shch'$ and $cjfj'$. And since the angle hch' is supplementary to both hsh' and jcj', therefore angles hsh' and jcj' are equal; as also angles jfj', and hch'; $ch=jf$; $sh=cj$, and $hh'=jj'$; $xs=cz$; and $zf=cx$.

Theorem 2. *The right lines, joining the foci of conjugate hyperbolas, pass through the intersections of the conjugate directrices; and cut off, upon each directrix, a portion* [ll', *and* ll''] *which is double its focal distance* [*i. e.*, $ll'=2xs$; *and* $ll'=2zf$].

Let the two directrices [hh' and jj'] intersect in l. Join sl and fl. Then, since $cxlz$ is a parallelogram; $lx=cz=xs$; and $zl=cx=zf$. Therefore the right angled triangles lxs and fzl are also isosceles, and the angles xsl, xls, zlf, and zfl equal to each other; and each half of a right angle.

Therefore slf is one right line, joining the two foci s and f and passing through the intersection (l) of the conjugate directrices. While ll' equals twice lx, and therefore twice xs, and similarly ll'' equals twice lz, or twice zf.

Theorem 3. *If P be the point where the hyperbola is cut by the directrix of its conjugate, then P lies on a circle, whose diameter is the focal distance ss'* [*i. e.*, $CP=Cs=Cf$ *in Fig. 1*], *while its tangent and produced normal, respectively, pass through the foci of the conjugate hyperbola.*

Let the conjugate directrix jj' cut the curve in P. Join CP, and draw Pt', and Pt, the tangent, and produced normal of P; cutting the minor axis in t' and t.

Then, Cz being the abscissa of P on the minor axis, $Cz.Ct'=CB^2$. But $Cz.Cf$ also equals CB^2. Hence Cf and Ct' are equal; and t' must coincide with f.

The angle $t'Pt$ being right, t must similarly coincide with f. And the circle, whose diameter is the focal distance ff', must therefore pass through P, and $CP=cf=cs$.

And similarly, if p' be the point of intersection of hh' with the conjugate curve, $Cp'=Cf=Cs$; and the tangent and produced normal of p' will pass through s' and s.

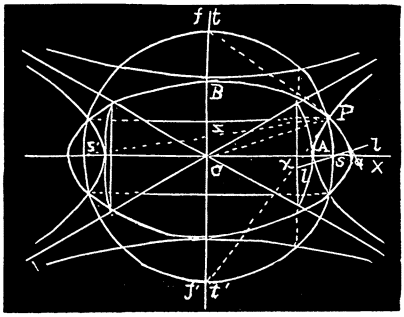
Corollary A. Therefore any point P, which lies on a hyperbola, equidistant from the center (c) with the focus (s), must have Cz for its abscissa on the minor axis; and f', f, the conjugate foci, for the intersections with the minor axis of its tangent and produced normal, respectively.

Corollary B. And so again, if the tangent and produced normal of any point P on a hyperbola, pass through the conjugate foci of the curve, then such point is equidistant from the center C with the foci s, s', f, and f'; and has the distance of the conjugate directrix from the center [*i. e.*, Cz] for its abscissa on the minor axis.

Theorem 4. *If two confocal conics—an ellipse and an hyperbola—be drawn, so that sx of the one equals in magnitude sx of the other, then their right angled intersections lie upon a circle, whose diameter is the inter-focal distance [i. e., $CP=CS=Cf$], coinciding, therefore, with the points of intersection of the hyperbola with its four conjugate directrices, and the minor axis [$2CB$] of these two confocals is common.*

At P, the point of right angled intersection of the two confocals, draw Pt', tangent at P to the hyperbola, intersecting its minor axis in t', and its directrix in l. And draw Pt, tangent at P to the ellipse, intersecting its minor axis in t, and its directrix in l'. Join sl, sl', sP, $s'P$, and CP.

Then, since in any conic curve the line drawn to the focus s from the intersection of any tangent with the directrix is perpendicular to the line of focal distance [sP] of such tangent, both lsP and $l'sP$ are right angles, lsl' one right line.



Then, the angles xsl and Xsl' being equal; and sx equalling sX, the right angled triangles sxl end sXl' are equal in all respects, and $sl=sl'$.

Therefore, again, the right angled triangles sPl and sPl' are also equal, and the angles sPl and sPl' equal, each to the other. Thus, the right angle lPl' has been bisected by sP; and the angle sPl equals half a right angle; and its double sPs' one right angle. Then the circle, whose diameter is the inter-focal distance ss', must pass through P, and $CP=Cs=Cf$.

But such a point, as the corollary to the last theorem [3. Corollary A] shows, must coincide with the intersection of the hyperbola with its conjugate directrix [jj']; and its tangent and produced normal must pass through the foci [ff'] of the conjugate hyperbola. Therefore cz must be the abscissa of P on the common minor axis; cf' the intercept on the minor axis of the tangent at P on the hyperbola, so that t' and f' coincide, and cf the intercept of the produced normal at P to the hyperbola, or tangent to the ellipse; so that t and f coincide.

Then, since the proportion $Ct'.Cn=BC^2$ [Cn being the abscissa, and Ct' the intercept of the tangent on the minor axis] belongs to the general conic; and since $Ct'=Ct=Cf$; and $Cn=Cz$ for both the confocals; therefore the third magnitude CB^2 must also be fixed and equal in both confocal curves; and $2CB$ be the common minor axis of ellipse and hyperbola.

NOTE ON THE EXPANSION OF DEVERTEBRATE DETERMINANTS.

By ORLANDO S. STETSON, Syracuse University.

In a paper entitled *"Co-axial Minors of a Determinant of the Fourth Order," Dr. Muir obtains what might be regarded as a sort of converse of Cayley's Expansion Theorem, viz., a general law for the expansion of an invertebrate co-axial minor in terms of the vertebrate co-axial minors of a given determinant and the elements of the principal diagonal. Dr. Muir mentions here, also, the possibility of presenting this as the ultimate case of a more general law, namely, a law for the expansion of a devertebrated† determinant in terms of the elements of the principal diagonal of the given determinant and their co-axial minors.

My attention has been drawn by Dr. William H. Metzler to the desirability of obtaining, if possible, an explicit statement of such a general law.

The object of this paper is to show how such a general law may be deduced. A slight modification of method will present, algebraically, a general proof of Cayley's Expansion Theorem given in Muir's Theory of Determinants.‡

Let $D=|a_{ij}|$ represent a general determinant of order n and let $\triangle$ be a determinant of the following form:

$$\triangle=\begin{vmatrix} a_{11}+x_{11} & a_{12}\cdots\cdots a_{1k}\cdots\cdots a_{1n} \\ a_{21} & a_{22}+x_{22}\cdots a_{2k}\cdots\cdots a_{2n} \\ \vdots & \vdots \qquad\quad \vdots \qquad\quad \vdots \\ a_{k1} & a_{k2}\cdots\cdots a_{kk}+x_{kk}\cdots a_{kn} \\ a_{n1} & a_{n2}\cdots\cdots a_{nk}\cdots\cdots a_{nn}+x_{nn} \end{vmatrix}.$$

Expanding $\triangle$ as a determinant with binomial elements, the term independent of the variables $x_{11}, x_{22}, \ldots\ldots, x_{nn}$ will be the given determinant D.

The coefficients of the variables $x_{11}, x_{22}, \ldots\ldots, x_{nn}$, are, respectively, the principal minors of order $(n-1)$ which we shall denote by $\underset{1}{A}, \underset{2}{A}, \ldots\ldots, \underset{n}{A}$. Similarly, the coefficients of $x_{11}x_{22}, x_{11}x_{33}, \ldots\ldots, x_{n-1}x_n$, are, respectively, the principal minors of order $(n-2)$ which we designate by $\underset{12}{A}, \underset{13}{A}, \ldots\ldots, \underset{n-1,n}{A}$; etc.

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*Transactions of the Royal Society of Edinburgh, Vol. 39, No. 10.

†k elements of principal diagonal zero, $n-k$ not zero.

‡Muir, p. 85.